
CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2016 Half-Yearly Examination

FORM VI

MATHEMATICS EXTENSION 1

Tuesday 1st March 2016

General Instructions

- Writing time — 1 hour and 30 minutes
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 55 Marks

- All questions may be attempted.

Section I – 7 Marks

- Questions 1–7 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 48 Marks

- Questions 8–11 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eight.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 111 boys

Examiner
DWH

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

1

The value of $\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ is:

(A) $\dfrac{\pi}{2}$

(B) $\dfrac{\pi}{3}$

(C) $\dfrac{\pi}{4}$

(D) $\dfrac{\pi}{6}$

QUESTION TWO

1

Find $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx\,.$

(A) $\sin^{-1}\left(\dfrac{x}{5}\right)+C$

(B) $\dfrac{1}{5}\sin^{-1}\left(\dfrac{x}{5}\right)+C$

(C) $\sin^{-1}(5x)+C$

(D) $\dfrac{1}{5}\sin^{-1}(5x)+C$

Exam continues next page ...

QUESTION THREE

1

Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x \, dx$.

(A) $\dfrac{1}{\sqrt{3}}$

(B) $\dfrac{2}{\sqrt{3}}$

(C) $\sqrt{3}$

(D) $2\sqrt{3}$

QUESTION FOUR

1

Which of the following is a point on the parabola $x^2 = 4ay$?

(A) $S\,(0, a)$

(B) $S'\,(0, -a)$

(C) $R\left(\dfrac{2a}{r}, \dfrac{a}{r^2}\right)$

(D) $Q\left(aq^2, 2aq\right)$

QUESTION FIVE

1

The area of a sector in a circle is given by the formula $A = \frac{1}{2}r^2\theta$. The radius r is fixed but the angle θ is increasing at a constant rate. The rate of change of A is:

(A) constant.

(B) zero.

(C) decreasing.

(D) increasing.

Exam continues overleaf ...

QUESTION SIX

1

The identity $\sin^{-1}(\sin x) = x$ is:

(A) false for all real x.

(B) true for all real x.

(C) false for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

(D) true for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

QUESTION SEVEN

1

Which of the following expressions is identical to $\sin\left(\dfrac{3\pi}{2} - x\right)$?

(A) $\cos\left(\dfrac{3\pi}{2} + x\right)$

(B) $\sin\left(\dfrac{3\pi}{2} + x\right)$

(C) $-\cos\left(\dfrac{3\pi}{2} + x\right)$

(D) $-\sin\left(\dfrac{3\pi}{2} + x\right)$

End of Section I

Exam continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 3\sin 2x\,dx$. **2**

(b) Find $f'(x)$ given $f(x) = \sin^2 x$. **1**

(c) Solve the equation $2\cos^2\theta + \sin\theta = 1$ for $0 \le \theta \le 2\pi$. **3**

(d) Consider the function $y = 3\sin^{-1}\left(\dfrac{x}{2}\right)$.

(i) Sketch the graph of the function. **2**

(ii) Find the gradient of the curve at the point where $x = 1$. **2**

(e) Find the acute angle between the lines with equations $y = -\frac{5}{2}x + 2$ and $y = -\frac{2}{3}x + 5$. Express your answer correct to the nearest minute. **2**

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) A spherical balloon is being inflated with air such that the volume V of the balloon increases at a constant rate of $10\,\text{cm}^3$ per second.

The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

(i) Find an expression for the rate of change of the radius r of the balloon. **2**

(ii) What is the radius at the instant when the radius is increasing at $0{\cdot}2\,\text{cm}$ per second? Give your answer correct to the nearest millimetre. **1**

(b) Find the general solution to the equation $\sin x = \dfrac{1}{\sqrt{2}}$. Give your solution in radians in exact form. **2**

(c) Consider the equation $3\cos x + 4\sin x = 4$.

(i) Write $3\cos x + 4\sin x$ in the form $R\cos(x - \alpha)$, where $0 < \alpha < 90^\circ$ and $R > 0$. Give α correct to the nearest degree. **2**

(ii) Hence solve $3\cos x + 4\sin x = 4$ in the domain $0^\circ \le x \le 360^\circ$. Give your solutions correct to the nearest degree. **2**

(d) Let $t = \tan\left(\dfrac{7\pi}{8}\right)$.

(i) Using a t-substitution for the expression $\tan\left(\dfrac{7\pi}{4}\right)$, or otherwise, show that **1**

$$t^2 - 2t - 1 = 0\,.$$

(ii) Hence find the exact value of $\tan\left(\dfrac{7\pi}{8}\right)$. **2**

Exam continues next page ...

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Use the properties of logarithms to write $\log_e\left(\dfrac{(x+1)^2}{2x}\right)$ without fractions. **1**

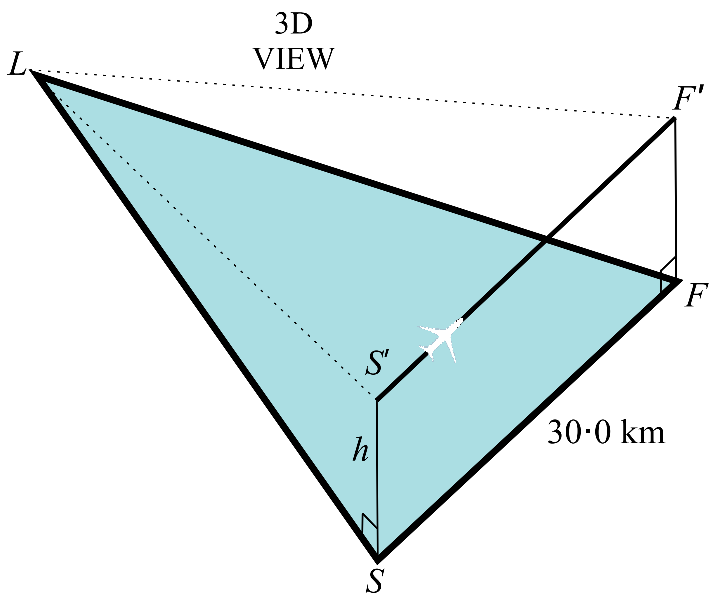
(ii) Find the coordinates of the stationary point on the curve $y = \log_e\left(\dfrac{(x+1)^2}{2x}\right)$. **2**

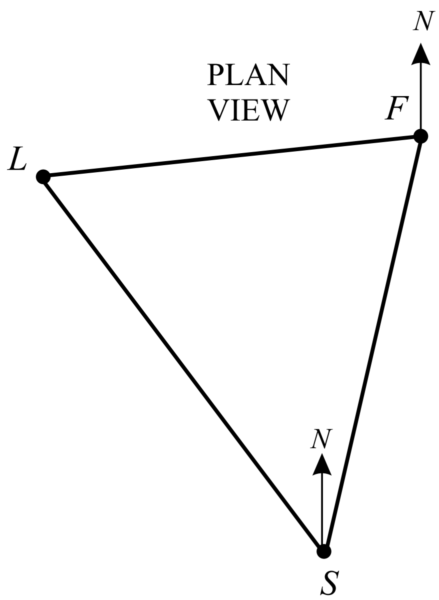
(b) Use Mathematical Induction to prove that for all positive integers n, **3**

$$(1\times 4) + (2\times 5) + (3\times 6) + \cdots + (n\times(n+3)) = \frac{1}{3}n(n+1)(n+5)\,.$$

(c) The parabola $x^2 = 6y$ intersects with the line $y = 2 - x$ at two points, A and B. The tangents to the parabola at A and B intersect at P. Find the co-ordinates of P. (You may use the chord of contact formula which is on the Reference Sheet.) **2**

(d)





A pilot flew a plane over a flat region at a constant altitude h. When she was at point S' vertically above point S, she observed a landmark L on the ground on a bearing of $323°T$ at an angle of depression of $7°$. After flying 30·0 kilometres on a bearing of $013°T$ she arrived at point F' vertically above F. She observed the same landmark L on a bearing of $264°T$.

(i) Show that $\angle FLS = 59°$. **1**

(ii) Find LS in terms of h. **1**

(iii) Find her altitude h. Give your answer correct to the nearest 100 metres. **2**

QUESTION ELEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) Suppose $P(2ap, ap^2)$ is a point on the parabola $x^2 = 4ay$.

(i) Show that the equation of the normal to the parabola at the point P is $x + py = 2ap + ap^3$. **2**

(ii) The normal at P cuts the y-axis at Q. The point R moves such that Q is the midpoint of RP. Show that the co-ordinates of R are $(-2ap, 4a + ap^2)$. **2**

(iii) Find the Cartesian equation of the locus of R as P varies. **2**

(iv) Describe the locus of R as a transformation of the original parabola $x^2 = 4ay$. **1**

(b) The curve $y = -\log_e(x + 1)$, the y-axis and the line $y = 1$ enclose a region.

(i) Sketch this region, clearly showing any intercepts with the axes and asymptotes. **2**

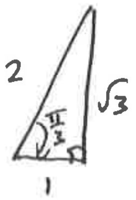
(ii) This region is rotated around the y-axis to form a solid of revolution. Find the volume of this solid. **3**

End of Section II

END OF EXAMINATION

2016 3U HALF-YEARLY FORM VI SOLUTIONS

(1)

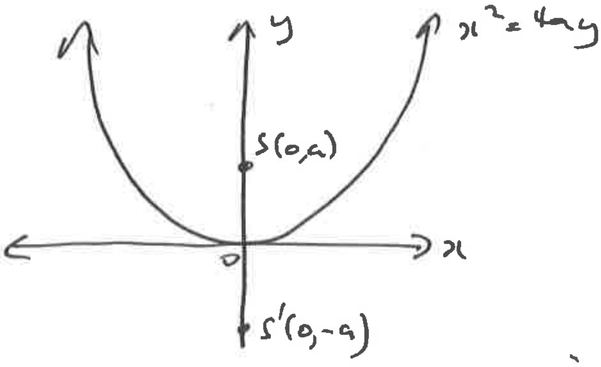


$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ (B)

(2) $\int \frac{1}{\sqrt{25-x^2}}\,dx = \sin^{-1}\left(\frac{x}{5}\right) + C$ (A)

(3) $\int_0^{\pi/3} \sec^2 x\,dx = \Big[\tan x\Big]_0^{\pi/3} = \tan\frac{\pi}{3} - 0 = \sqrt{3}$ (C)

(4)



$R:\ x^2 = \left(\frac{2a}{r}\right)^2 = \frac{4a^2}{r^2} = 4a\left(\frac{a}{r^2}\right)$

$= 4ay$ ✓

$Q:\ x^2 = (aq^2)^2 \neq 4a(2aq)$ ✗

$\therefore$ (C)

(5) r is constant

$A = \frac{1}{2}r^2\theta$

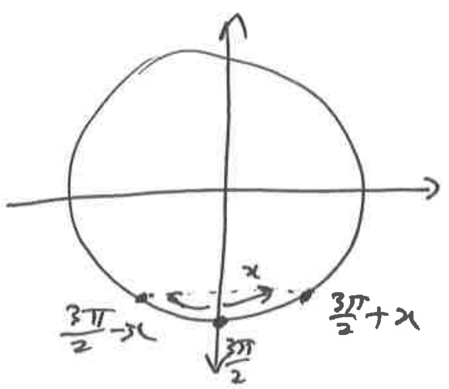
$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}$, which is constant (A)

(6) $\sin^{-1}(\sin x) = x$ range ~~range~~ ~~domain~~ of $\sin^{-1}(q)$ is $-\frac{\pi}{2} \le q \le \frac{\pi}{2}$

$\therefore$ range of RHS is also $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$\therefore$ (D)

(7)



$\frac{3\pi}{2} - x$ and $\frac{3\pi}{2} + x$ have equal <u>sine</u> values

$\therefore$ (B)

8 (a) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 3\sin 2x\,dx = 3\left[\frac{-\cos 2x}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$ ✓

$= -\frac{3}{2}\left[\cos\frac{2\pi}{3} - \cos\frac{\pi}{3}\right]$

$= -\frac{3}{2}\left[-\frac{1}{2} - \frac{1}{2}\right]$ ✓

$= \frac{3}{2}.$

(b) $f(x) = \sin^2 x$

$f'(x) = 2\sin x\cos x$ (Chain rule) ✓

(c) $2\cos^2\theta + \sin\theta = 1$ $\quad 0 \le \theta \le 2\pi$

$2(1-\sin^2\theta) + \sin\theta = 1$ ✓

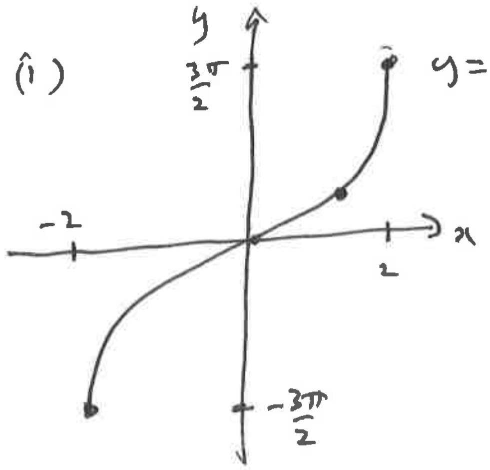
$0 = 2\sin^2\theta - \sin\theta - 1$

$= (2\sin\theta + 1)(\sin\theta - 1)$ ✓

$\therefore \sin\theta = -\frac{1}{2}$ or 1 ✓

$\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$

(d) (i)



✓ ✓

$-1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$

$-\frac{\pi}{2} \le \sin^{-1}\left(\frac{x}{2}\right) \le \frac{\pi}{2}$

$-\frac{3\pi}{2} \le 3\sin^{-1}\left(\frac{x}{2}\right) \le \frac{3\pi}{2}$

(ii) $y = 3\sin^{-1}\left(\frac{x}{2}\right)$

$\frac{dy}{dx} = 3 \cdot \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2}$ ✓

when $x = 1$

$m_{tangent} = \frac{3}{2\sqrt{\frac{3}{4}}} = \sqrt{3}$ ✓

8 (e) $y = -\frac{5}{2}x + 2$ $\quad m_1 = -\frac{5}{2}$

$y = -\frac{2}{3}x + 5$ $\quad m_2 = -\frac{2}{3}$

Let $\theta = \angle$ betwen the two lines

$$\tan\theta = \left| \frac{-\frac{5}{2} - \left(-\frac{2}{3}\right)}{1 + \left(-\frac{5}{2}\right)\left(-\frac{2}{3}\right)} \right|$$

$$= \frac{11}{16}$$

$$\theta \approx 34^\circ 31'$$

$\therefore$ The angle between the two lines is $\approx 34^\circ 31'$

(9) (a) (i) $\frac{dV}{dt} = +10 \text{ cm}^3/\text{s}$

$$V = \frac{4}{3}\pi r^3 \text{ cm}^3$$

$$\frac{dV}{dr} = 4\pi r^2 \text{ cm}^3/\text{cm}$$

$$\frac{dr}{dt} = \frac{dr}{dV} \cdot \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \cdot 10 \text{ cm/s}$$

$$= \frac{5}{2\pi r^2} \text{ cm/s}$$

(ii) when $\frac{dr}{dt} = 0.2$ cm/s

$$0.2 = \frac{5}{2\pi r^2}$$

$$r^2 = \frac{25}{2\pi}$$

$$r = \sqrt{\frac{25}{2\pi}} \approx 2.0 \text{ cm}$$

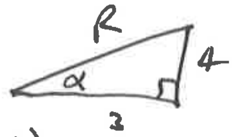
(9) (b) $\sin x = \frac{1}{\sqrt{2}}$ ✓ ✓

$x = \frac{\pi}{4}(-1)^n + \pi n$ where n is an integer

$\underline{\underline{\text{or}}}$ $x = \frac{\pi}{4} + 2\pi n$ or $\frac{3\pi}{4} + 2\pi n$ where n is an integer

(c)(i) $3\cos x + 4\sin x = R\left(\frac{3}{R}\cos x + \frac{4}{R}\sin x\right)$

so if $\frac{3}{R} = \cos\alpha$ and $\frac{4}{R} = \sin\alpha$



$R = \sqrt{3^2+4^2} = 5$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right) \approx 53°$ (nearest degree)

$\therefore 3\cos x + 4\sin x \approx 5\cos(x - 53°)$ ✓ ✓

(ii) $3\cos x + 4\sin x = 4$

$5\cos(x-53°) = 4$

$\cos(x-53°) = \frac{4}{5}$ $\qquad -53° \leq x - 53° \leq 307°$

$\therefore x - 53° \approx -37°, 37°$ ✓

$\underline{x \approx 16°, 90°}$ ✓

(d) $t = \tan\left(\frac{7\pi}{8}\right)$

(i) $\tan\left(2\times\frac{7\pi}{8}\right) = \frac{2\tan\left(\frac{7\pi}{8}\right)}{1-\tan^2\left(\frac{7\pi}{8}\right)}$ ✓

$-1 = \frac{2t}{1-t^2}$

$\therefore t^2 - 2t - 1 = 0$

(ii) $t = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$ ✓

but $\tan\left(\frac{7\pi}{8}\right) < 0$ (2nd Quadrant)

$\therefore \underline{t = 1 - \sqrt{2}}$ ✓

(10) (a) (i) $\ln\left(\frac{(x+1)^2}{2x}\right) = 2\ln(x+1) - \ln 2x$ ✓

$= 2\ln(x+1) - \ln x - \ln 2$

(ii) $\frac{dy}{dx} = \frac{2}{x+1} - \frac{1}{x} - 0$ ✓

st. pt. at $\frac{dy}{dx} = 0 = \frac{2}{x+1} - \frac{1}{x}$

$\frac{1}{x} = \frac{2}{x+1}$

$x+1 = 2x$

$\underline{x = 1}$

$y = \ln\left(\frac{2^2}{2}\right) = \ln 2$ ✓

$\therefore$ St. pt. at $(1, \ln 2)$

(b) <u>RTP:</u> $1\times4 + 2\times5 + 3\times6 + \cdots + n(n+3) = \frac{1}{3}n(n+1)(n+5)$

<u>PROOF:</u> if $n=1$, LHS $= 4$

RHS $= \frac{1}{3}\times1\times2\times6 = 4 =$ LHS ✓

$\therefore$ The result is true for $n=1$.

Let's assume the result is true for some integer k.

i.e. $1\times4 + 2\times5 + 3\times6 + \cdots + k(k+3) = \frac{1}{3}k(k+1)(k+5)$ ✓

$\Rightarrow 1\times4 + 2\times5 + \cdots + k(k+3) + (k+1)(k+4)$

$= \frac{1}{3}k(k+1)(k+5) + (k+1)(k+4)$

$= \frac{1}{3}(k+1)\left[k(k+5) + 3(k+4)\right]$

$= \frac{1}{3}(k+1)\left[k^2 + 8k + 12\right]$

$= \frac{1}{3}(k+1)(k+6)(k+2)$ ✓

$= \frac{1}{3}(k+1)\left[(k+1)+1\right]\left[(k+1)+5\right]$

$\therefore$ If the result is true for k, then it's also true for $k+1$.

Since the result is true for 1, by the principle of Mathematical Induction, it's true for all positive integers

10

(c) From the description, the line $y = 2 - x$ is the chord of contact on the parabola from the point P.

Let P be (x_0, y_0)

Eqn of chord of contact is

$$x x_0 = 2a(y + y_0)$$

$x^2 = 6y$ has a focal length of $\frac{6}{4} = \frac{3}{2}$

$\therefore$ Eqn is

$$x x_0 = 3(y + y_0)$$

$$\frac{x x_0}{3} - y_0 = y$$

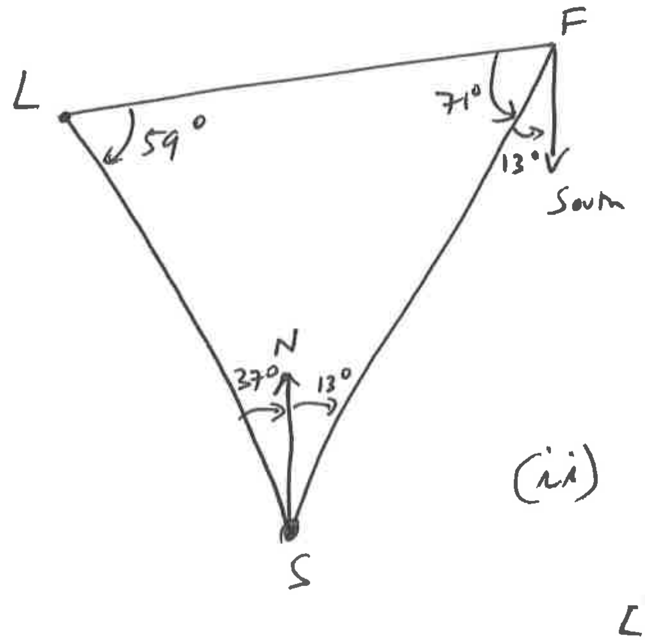
which is the same line as $y = 2 - x$

$\therefore \frac{x_0}{3} = -1$ and $y_0 = -2$

$x_0 = -3$

$\therefore$ The point P is $(-3, -2)$.

(d)



(i) $\angle NSL = 360° - 323° = 37°$

$\angle LSF = 37° + 13° = 50°$

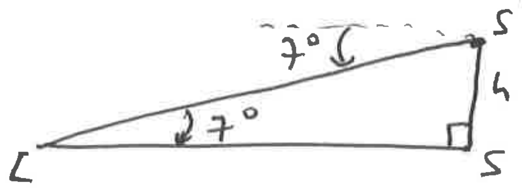
$\angle SF(\text{South}) = 13°$ (alt $\angle$s)

$\angle SFL = 264° - 180° - 13° = 71°$

$\therefore \angle FLS = 180° - 50° - 71°$ ($\angle$ sum Δ)

$= 59°$

(ii)



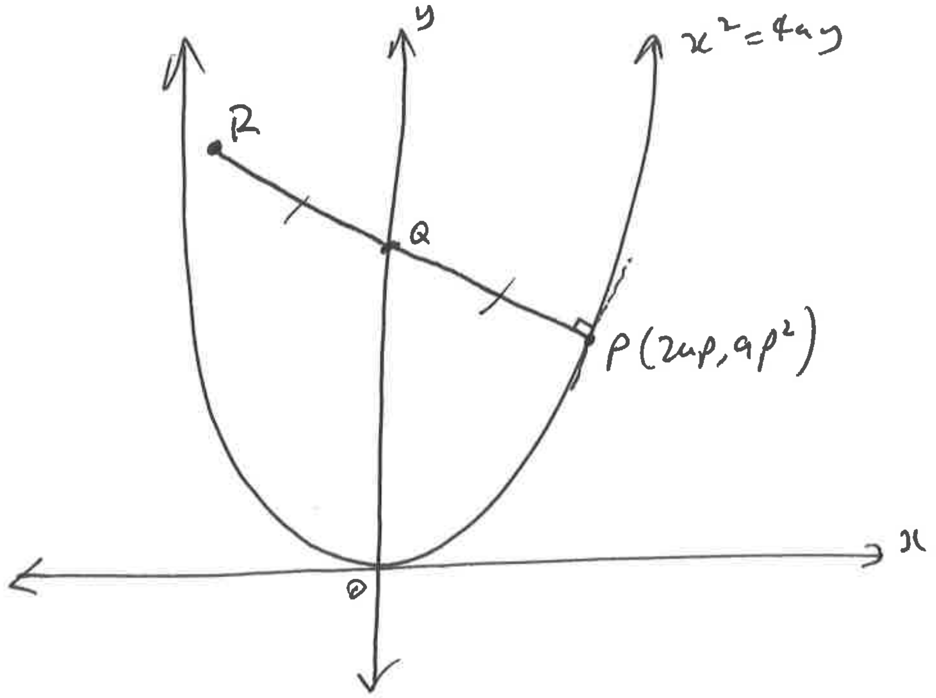
$$\frac{h}{SL} = \tan 7° \Rightarrow SL = \frac{h}{\tan 7°}$$

(iii) SINE RULE IN Δ FLS :

$$\frac{\frac{h}{\tan 7°}}{\sin 71°} = \frac{30.0\text{ km}}{\sin 59°}$$

$$h = \frac{30 \tan 7° \sin 71°}{\sin 59°}\text{ km} \approx 4.01\text{ km}$$

11 (a)



(i) $x^2 = 4ay$

$\frac{dy}{dx} = \frac{2x}{4a}$ at P, $x = 2ap$

$\therefore m_{\text{tangent at } P} = \frac{2(2ap)}{4a} = p$

$\therefore m_{\text{normal at } P} = -\frac{1}{p}$

$\therefore$ EQN normal:

$y-(ap^2) = -\frac{1}{p}(x-2ap)$

$py - ap^3 = -x + 2ap$

$x + py = ap^3 + 2ap$

(ii) Q is the point on $x+py = ap^3+2ap$ where $x=0$

$\therefore py = ap^3 + 2ap$

$y = ap^2 + 2a$ (assuming $p \neq 0$, where Q would not be defined)

$\therefore$ Q is $(0, ap^2+2a)$ and P is $(2ap, ap^2)$

so since Q is the midpoint of $R(x_0, y_0)$

$$\left(0, ap^2+2a\right) = \left(\frac{x_0+2ap}{2}, \frac{y_0+ap^2}{2}\right)$$

$x_0 = -2ap$, $y_0 + ap^2 = 2ap^2 + 4a$

$y_0 = ap^2 + 4a$

$\therefore$ R is $(-2ap, ap^2+4a)$

(iii) Locus of R: $x^2 = 4a^2p^2 = 4a(ap^2)$

$= 4a(y-4a)$

(iv) The locus of R is the translation of $x^2 = 4ay$ up by $4a$ units, but not including the vertex $(0, 4a)$ since $p \neq 0$.

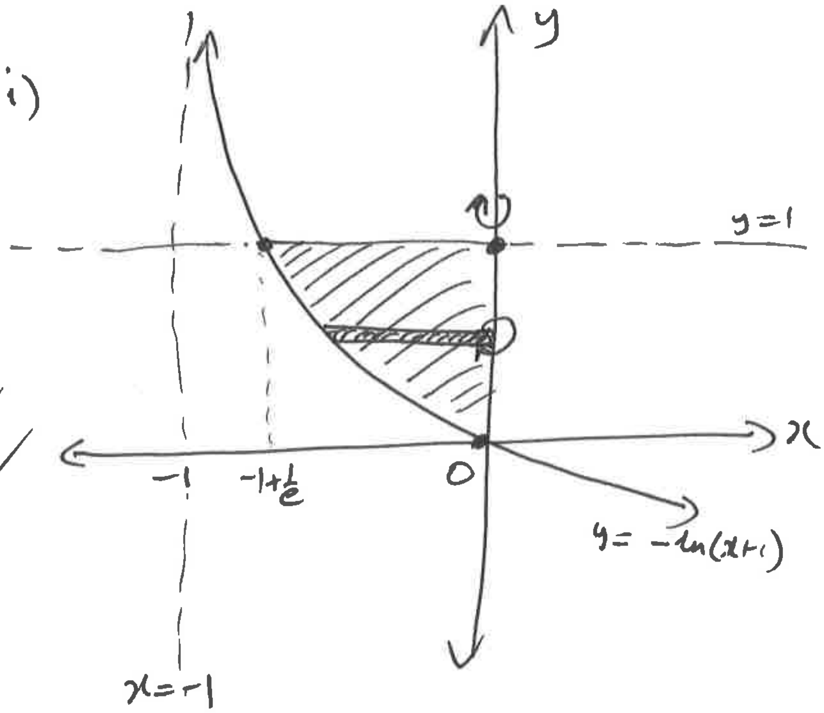
(11) (b) $y = -\ln(x+1)$

$e^{-y} = x+1$

$x = e^{-y} - 1$

when $y=1$, $x = \frac{1}{e} - 1$

(i)



(ii) $V = \pi \int_{y=0}^{y=1} (x^2)\, dy$

$= \pi \int_0^1 (e^{-y} - 1)^2\, dy$

$= \pi \int_0^1 (e^{-2y} - 2e^{-y} + 1)\, dy$

$= \pi \left[\frac{e^{-2y}}{-2} - \frac{2e^{-y}}{-1} + y \right]_0^1$

$= \pi \left[\left(-\frac{e^{-2}}{2} + \frac{2}{e} + 1 \right) - \left(-\frac{1}{2} + 2 + 0 \right) \right]$

$= \pi \left(\frac{-1}{2e^2} + \frac{2}{e} - \frac{1}{2} \right)$

$= \frac{\pi}{2e^2} (-e^2 + 4e - 1)$ units^3 (≈ 0.528)